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The stability of a two-dimensional stagnation flow to three-dimensional disturbances

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Experiments have shown that the two-dimensional flow near a forward stagnation line may be unstable to three-dimensional disturbances. The growing disturbance takes the form of secondary vortices, i.e. vortices more or less parallel to the original streamlines. The instability is usually confined to the boundary layer and the spacing of the secondary vortices is of the order of the boundary-layer thickness. This situation is analysed theoretically for the case of infinitesimal disturbances of the type first studied by Görtler and Hämmerlin. These are disturbances periodic in the direction perpendicular to the plane of the flow, in the limit of infinite Reynolds number. It is shown that the flow is always stable to these disturbances.

1. Introduction

The idea that the flow near a two-dimensional stagnation point may be unstable was first put forward by Görtler (1955). The mechanism which will create the instability, if any, is the centrifugal instability which was investigated by Taylor (1923) for the case of Couette motion and by Görtler (1940) for the case of boundary-layer flow on a concave wall, and which may occur (as in the present situation) in any flow with curved streamlines when the fluid on the inside of the curve is moving sufficiently rapidly compared with the fluid on the outside.

When this form of instability occurs the growing disturbance takes the form of a secondary motion whose vortex lines more or less coincide with the undisturbed streamlines. In many cases the secondary motion will cease to grow, at a certain amplitude, and the ultimate state of the flow will comprise the original basic flow together with a steady secondary motion in the form of cells spaced transversely to the basic flow. In some of the experiments on stagnation-point flows (to be described later) the cells grow spatially (i.e. the amplitude of the secondary motion grows) in the streamwise direction.

Görtler (1955) obtained the linearized disturbance equations in the case of two-dimensional stagnation flow against an infinite plane and the equations were studied in detail by Hämmerlin (1955). The disturbance quantities are assumed, in the usual way, to be proportional to $\exp(\beta t + i\alpha z)$, where z is measured normal to the plane of the flow, and the problem is to determine β, given α. It is customary to focus attention on the case $\beta = 0$, when α should be determined as an eigenvalue of the resulting system. However Hämmerlin (1955) concluded that the eigenvalues form a continuous spectrum, $0 < \alpha < 1$.

The flow studied by Görtler and Hämmerlin is of course the appropriate model for flow near the forward stagnation line on a two-dimensional blunt body, when the Reynolds number R is large. When the equations are written in boundary-layer variables R no longer appears, leaving α and β as the only explicit parameters. There is, therefore, no possibility of obtaining the usual curve of neutral stability in the α, R plane; only the asymptote as $R \to \infty$ can be found.

Hämmerlin's result seems unsatisfactory because a unique eigenvalue would be expected, and the problem was re-examined by Kestin & Wood (1970), who argued (correctly in our opinion) that the root cause of the trouble lies in the over-idealization of the problem as considered by Görtler and Hämmerlin. There is no natural length scale in this model and one suspects that, as a result, vital information about the structure of the flow at infinity has been lost. In order to retain some of the geometry of a realistic flow, Kestin & Wood (1970) derived the disturbance equations for the flow near the forward stagnation point on a circular cylinder and argued that certain small terms, associated with the curvature of the wall, must be retained in order to obtain a unique eigenvalue. In this work it will be argued that the remedy given by Kestin & Wood is incorrect and that the correct solution is actually simpler; the equations of Görtler and Hämmerlin lead to a unique eigenvalue provided that the information at infinity is used to derive the correct boundary condition. In particular it will be shown that boundary curvature is irrelevant (by considering the flow past a flat plate of finite width set broadside on to the stream) and that the cylinder problem considered by Kestin & Wood, and indeed the corresponding problem for any blunt-nosed body, can be reduced for $R \to \infty$ to the flat-plate problem. [The problem referred to here is the determination of the eigenvalues of the disturbance equations; of course the spatial growth of the disturbance will be modified by the details of the boundary shape.]

The main purpose of the present work, then, is to present the numerical solution of the disturbance equations proposed by Görtler (1955) together with the more stringent boundary conditions at infinity, to be explained in the next section. It turns out that on setting $\beta = 0$ there is no eigenvalue α (this is the problem for which Hämmerlin (1955) obtained a continuous spectrum). This means that there is no neutral wave-number. The equations were then solved by restoring β to the equations and solving for β as an eigenvalue with α prescribed. The result is that $\beta < 0$ for all α, with the implication that the flow is stable.

There is, however, considerable experimental evidence that instability of this kind can occur. Experiments on two-dimensional flow past blunt bodies have been described by Kestin & Wood (1970), Brun, Diep & Kestin (1966), Colak-Antic (1971) and Hassler (1971); the results are summarized in figure 1. Other experiments, on wedges and cones, are reported by Görtler & Hassler (1973). It must be concluded that this instability is, as yet, not satisfactorily explained.

2. Flow past a transverse flat plate

We consider here the flow of a uniform stream of speed U_∞ past a flat plate of finite width $2d$ set symmetrically broadside on to the stream. Cartesian axes are fixed in the plate with the origin O at the centre, x measured along the plate in the plane of the flow and y normal to the plate and pointing upstream.

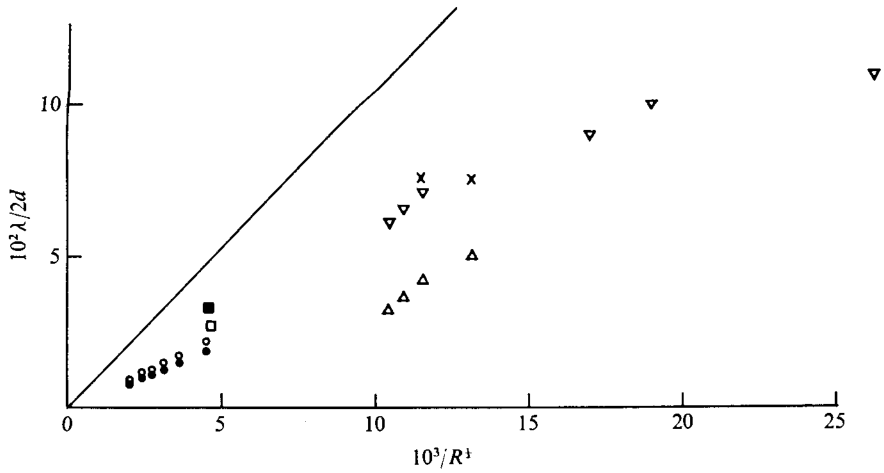

FIGURE 1. Experimental results. The dimensionless wavelength $\lambda/2d$ is plotted against $R^{-\frac{1}{2}}$. ○, ●, Kestin & Wood, lower and higher turbulence intensity; □, ■, Brun *et al.*, lower and higher turbulence intensity. Colak-Antic or Hassler: △, near the stagnation line; ▽, 4° from the stagnation line; ×, average values obtained from photographs. According to the theory these points should lie on straight lines through the origin. The straight line shown corresponds to the least-damped waves in the present theory, for which $\alpha = 0{\cdot}298$.

The first preliminary is to work out the boundary layer near the stagnation point (the origin). The potential flow past the plate is easily obtained and gives the dimensionless stream function near the stagnation point as

$$\psi = xy + 2x^3y. \tag{2.1}$$

Here terms in x^5, and y^3, etc. have been ignored. To obtain the boundary layer we set $\eta = R^{\frac{1}{2}}y$ and $\Psi = R^{\frac{1}{2}}\psi$; the equation and boundary conditions are satisfied by $\Psi = xF(\eta)$, where F satisfies

$$F''' + FF'' - F'^2 + 1 = 0 \tag{2.2}$$

together with $F(0) = F'(0) = 0$ and $F'(\infty) = 1$, in the usual way. Here we are also ignoring the term in x^3 in (2.1); this will be considered later.

The velocity components in the boundary layer of this basic flow are $(U, V, 0)$, where $U = \Psi_\eta$ and $V = -R^{-\frac{1}{2}}\Psi_x$. To obtain the stability problem we study small perturbations of these, i.e. we let the velocity be $(U+u, V+v, w)$ and linearize the equations. This is a familiar process and we shall not give the details here. It is necessary to introduce a scaled co-ordinate transverse to the flow, $Z = R^{\frac{1}{2}}z$, and scaled disturbance velocity components $w^* = R^{\frac{1}{2}}w$ and $v^* = R^{\frac{1}{2}}v$. It is also appropriate to assume that all disturbance quantities are proportional to $\exp(i\alpha Z + \beta t)$. The introduction of Z is equivalent to rescaling the wavenumber α and indicates that the expected transverse wavelength is of the same order as the boundary-layer thickness. The velocity components are scaled in order to keep them of the same order as the basic boundary-layer velocities.

The continuity equation becomes

$$\frac{\partial u}{\partial x} + \frac{\partial v^*}{\partial \eta} + \frac{\partial w^*}{\partial Z} = 0 \tag{2.3}$$

and the linearized equations are

$$\beta u+\Psi_{\eta x}u+\Psi_{\eta}\frac{\partial u}{\partial x}+\Psi_{\eta\eta}v^*-\Psi_{x}\frac{\partial u}{\partial \eta}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{1}{R}\frac{\partial^2 u}{\partial x^2}+\frac{\partial u^2}{\partial \eta^2}-\alpha^2 u, \tag{2.4}$$

$$\beta v^*-\Psi_{xx}u+\Psi_{\eta}\frac{\partial v^*}{\partial x}-\Psi_{x\eta}v^*-\Psi_{x}\frac{\partial v^*}{\partial \eta}=-\frac{R}{\rho}\frac{\partial p}{\partial \eta}+\frac{1}{R}\frac{\partial^2 v^*}{\partial x^2}+\frac{\partial^2 v^*}{\partial \eta^2}-\alpha^2 v^*, \tag{2.5}$$

$$\beta\omega^*+\Psi_{\eta}\frac{\partial w^*}{\partial x}-\Psi_{x}\frac{\partial w^*}{\partial \eta}=-\frac{R}{\rho}i\alpha p+\frac{1}{R}\frac{\partial^2 w^*}{\partial x^2}+\frac{\partial^2 w^*}{\partial \eta^2}-\alpha^2 w^*. \tag{2.6}$$

Here p denotes the disturbance pressure. The somewhat unusual appearance of this set of equations has been retained in order to explain one or two points concerning the terms which are negligible. First it is apparent that p must be rescaled by a factor R, which will cause it to disappear from (2.4). Next, since x has not been rescaled the x derivatives on the right-hand sides are small. It might be thought, therefore, that the neglect of these terms would be in order when $x = O(1)$ but not when $x = O(R^{-\frac{1}{2}})$, necessitating a separate asymptotic theory in this region. However this turns out to be unnecessary because the solution to be obtained for $x = O(1)$ vanishes identically when substituted into the neglected terms and is thus uniformly valid down to $x = 0$. This form of solution, proposed by Görtler (1955), is

$$u = xu_0(\eta), \quad v^* = v_0(\eta), \quad w^* = w_0(\eta), \quad p = (\rho/R)\,p_0(\eta) \tag{2.7}$$

and each expression is of course to be multiplied by $\exp(i\alpha Z+\beta t)$. There results the following system of equations:

$$u_0+v_0'+i\alpha\omega_0 = 0, \tag{2.3a}$$

$$u_0''+Fu_0'-(\alpha^2+\beta+2F')\,u_0 = F''v_0, \tag{2.4a}$$

$$v_0''+Fv_0'-(\alpha^2+\beta-F')\,v_0 = p_0', \tag{2.5a}$$

$$w_0''+Fw_0'-(\alpha^2+\beta)\,\omega_0 = i\alpha p_0. \tag{2.6a}$$

It is interesting to obtain the equation for the secondary vorticity, i.e. the x component of the vorticity. This is given by $\Omega_0 = w_0'-i\alpha v_0$ and satisfies the equation

$$\Omega_0''+F\Omega_0'-(\alpha^2+\beta-F')\,\Omega_0 = 0. \tag{2.8}$$

This system was obtained by Kestin & Wood (1970), together with a number of small terms associated with the curvature of the boundary.

On the solid boundary we have the usual condition

$$u_0 = v_0 = w_0 = 0 \quad \text{on} \quad \eta = 0 \tag{2.9}$$

and we turn now to the crucial question of the boundary conditions at infinity. Görtler and Hämmerlin proposed that u_0, v_0 and w_0 are $o(1)$ as $\eta \to \infty$ (i.e. they merely tend to zero), and Hämmerlin deduced that when $\beta = 0$ the eigenvalues α form a continuous spectrum over $0 < \alpha < 1$. Kestin & Wood (1970) argued that the reason why a unique eigenvalue is not obtained lies in the fact that the coefficient function F appearing here is only an approximation and that the retention of certain small terms in the system will alter its character at infinity in such a way as to produce a discrete spectrum. They proposed that letting $R \to \infty$ with $\eta = R^{\frac{1}{2}}y$ fixed, which gives the system (2.3a)–(2.6a), is not strictly valid because the terms so neglected are important outside the boundary layer, i.e. as $\eta \to \infty$. While in a sense this is true, as a remedy for the difficulty before us it is misconceived. What is developed here is an asymptotic theory, as $R \to \infty$, which will give the first term in the asymptotic expansion of the eigenvalue β as $R \to \infty$. The system (3.2*a*)–(3.5*a*) is the 'inner' set of equations valid for fixed η and should be complemented by another set, the outer set, valid for fixed y as $R \to \infty$, and the boundary condition at infinity on the inner set should be obtained by matching. (We shall in fact avoid this procedure, which seems in any case to be intractable.) But it is quite wrong in general to attempt to derive the outer system from the inner system merely by altering the coefficient functions to some composite approximation; it must be obtained from the full equations.

Kestin & Wood concentrated on the vorticity equation (2.8), and attempted to derive boundary conditions on it which would produce a discrete eigenvalue spectrum. They appear in fact to have derived the boundary condition on the wall from the equation itself, a procedure which cannot be correct, and although the wall condition can presumably be written in the form $\Omega'(0)+\gamma\Omega(0) = 0$ (as they conclude), since higher derivatives can be eliminated by means of the equation, the constant γ will depend on α, the eigenvalue, in a unknown way. For this reason alone the results from the spectral theory of singular differential operators quoted by them are inapplicable.

However it is the condition at infinity which is crucial; here Kestin & Wood recast (2.8) by replacing $F(\zeta)$ by the 'large constant' $R^{\frac{1}{2}}$. The resulting equation has a solution which is exponentially small at infinity and they used this by requiring that (2.8) should have this solution as its asymptotic form. However it is clear that (2.8) does not have a solution which meets this requirement and in any case the idea is misconceived; for if we revert to outer variables (y instead of η) in the coefficient we must do likewise in the derivatives, and there results [from (2.8)] $d(\Omega y)/dy-\alpha^2\Omega = 0$, the non-trivial solution of which is exponentially *large* as $y \to \infty$. The conclusion is that $\Omega = 0$ in the outer region, which is no more than the condition applied by Görtler and Hämmerlin. The point here is simply that the far field of the boundary-layer function F is the *near* field of the potential flow, and this increases linearly (or rather decreases as $y \to 0$) and is not approximately a constant.

Here it is proposed that the disturbance quantities should tend to zero exponentially, as $y \to \infty$, and several reasons will be given for this. First we note that the mainstream vorticity decays exponentially outside the boundary layer. We give below an argument to show why the instability can originate only in the region of appreciable mainstream vorticity, i.e. the boundary layer; once this has been done it is natural to require exponential decay of the disturbance vorticity because this can penetrate upstream of the boundary layer only by diffusion.

To see why the destabilizing forces are confined to the boundary layer we must reconsider the well-known result, due to Rayleigh, for flow with circular streamlines: that the flow is stable or unstable according as the square of the circulation increases or decreases outwards. Viscosity will act to damp out the disturbance and it is the balance between these two effects which was studied by Taylor (1923). (We emphasize that the destabilizing forces are inviscid in character; although it is well known that viscous forces may destabilize a parallel shear flow, that mechanism is quite different to the one being considered here.)

The generalization of Rayleigh's criterion to arbitrary two-dimensional inviscid flow was given by Scorer & Wilson (1963). A stream surface is supposed to suffer a disturbance in the form of a small corrugation in the spanwise direction; this disturbance is steady but may grow spatially in the streamwise direction. The principal result may be stated as follows. Let ϕ denote the angle between the disturbed stream surface and its undisturbed form, at any point, and let s denote arc length measured along the streamline at that point. Then

$$d^2\phi/ds^2 = (\kappa\omega/q)\sin\phi,$$

where κ is the streamline curvature, ω is the undisturbed vorticity and q is the speed. Thus the left-hand side is the rate of increase of secondary vorticity following a particle.

The equation shows that the generalization of Rayleigh's criterion is that the flow is stable or unstable according as $\kappa\omega$ is negative or positive. A small disturbance causes the vorticity vector to acquire first of all a component along the principal normal to the streamlines and then, when the streamlines are curved, a component along the tangent. If this component has the same direction as the vorticity of the original disturbance, the disturbance will tend to grow. As in the case of Couette flow, viscosity will tend to damp out the disturbance, so that it does not follow that $\kappa\omega > 0$ implies instability (although $\kappa\omega < 0$ implies stability); the conclusion we wish to draw is that the destabilizing forces are confined to the region where ω is appreciable, namely the boundary layer. (Note that the velocity gradient along the normal to the streamlines is not the same as the vorticity; this quantity is in fact appreciable outside the boundary layer.)

We have argued that the disturbance vorticity should be exponentially small outside the boundary layer; this enforces exponential decay of the velocity components as may be seen from the asymptotic forms given in the next section. A further, more mathematical, argument in favour of exponential decay is as follows. This is really a matching problem, and it is well known that exponential decay is almost invariably required in such problems, algebraic decay usually leading to inconsistencies at higher orders. In the present case it would be necessary at the next order to determine a potential function having variations in the z direction of length scale $R^{-\frac{1}{2}}$.† The formal difficulties prevent the solution being carried out to the point where inconsistencies actually appear (in the form of 'impossible' boundary-value problems) but the arguments in favour of exponential decay seem to be reasonably conclusive. Similar conclusions were reached by Kelly (1962), who considered an eigenvalue problem somewhat resembling the present one.

Finally we show that the requirement of exponential decay will very probably convert the continuous spectrum found by Hämmerlin into a discrete spectrum by considering the model eigenvalue problem

$$\frac{d^2\Omega}{d\eta^2}+\eta\frac{d\Omega}{d\eta}+\eta\Omega = 0, \quad \Omega(0) = 0, \quad \Omega(\infty) = 0.$$

This bears a close resemblance to (2.8) since $F \sim \eta$ at infinity. If we merely require $\Omega(\infty) = 0$ then any $s > 0$ is an eigenvalue since the solutions have the asymptotic forms η^{-s} and $\eta^{s-1}\exp(-\frac{1}{2}\eta^2)$. The requirement of exponential decay eliminates the first solution and now the condition at $\eta = 0$ produces a discrete spectrum, easily shown in in fact to consist of the even integers.

† The author is indebted to a referee for this remark.

The numerical solution proved to be a difficult and delicate matter and since the accuracy of this solution is the crux of the whole analysis it seems important to give some explanation of it. This is given in the next section together with some discussion of the results.

Finally in this section we show briefly how the flow past a circular cylinder (and hence obviously any blunt-nosed body) leads to the same eigenvalue problem. We take axes $Oxyz$ with O at the stagnation point, x measured round the cylinder, y normal to it and z parallel to the generators. (This is the system used by Kestin & Wood.) Thus these co-ordinates are related to cylindrical polars (r, θ, z) by $r = 1+y$ and $\theta = x$ in dimensionless form. The potential flow and the basic boundary layer are easily obtained as power series in x, and we shall merely note here that the first term in the boundary-layer series is the same as for the flat plate.

We now obtain the disturbance equations as in the previous section, introducing the same boundary-layer variables. The momentum equations, replacing (2.4) and (2.5), are

$$\beta u + \Psi_{\eta x} u + \Psi_\eta \frac{\partial u}{\partial x} + \Psi_{\eta\eta} v^* - \Psi_x \frac{\partial u}{\partial \eta} + R^{-\frac{1}{2}} (\Psi_\eta v^* - \Psi_x u) = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{1}{R}\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial \eta^2} - \alpha^2 u, \tag{2.10}$$

$$\beta v^* - \Psi_{xx} u + \Psi_\eta \frac{\partial v^*}{\partial x} - \Psi_{x\eta} v^* - \Psi_x \frac{\partial v^*}{\partial \eta} - 2R^{\frac{1}{2}} \Psi_\eta u = -\frac{R}{\rho}\frac{\partial p}{\partial \eta} + \frac{1}{R}\frac{\partial^2 v^*}{\partial x^2} + \frac{\partial^2 v^*}{\partial \eta^2} - \alpha^2 v^* \tag{2.11}$$

and the other equation is the same to leading order in R. Again certain small terms have been retained, which might become important when x is small, and a large number of terms of relative order $R^{-\frac{1}{2}}$ (to the largest retained) have been omitted. The centrifugal effects are contained in the last terms on the left-hand sides. Clearly the bracket containing two terms in (2.10) can safely be neglected. Considerably more alarming is the apparently large term in $R^{\frac{1}{2}}$ on the left of (2.11). However we may remove this term by rescaling x; that is, we put $\xi = R^{\frac{1}{2}} x$ and $u^* = R^{\frac{1}{2}} u$, then note that $\Psi_\eta = xF' = R^{-\frac{1}{2}}\xi F'$ and we see that this term is of relative order $R^{-\frac{1}{2}}$ and may be neglected. The equations are not quite the same as (2.4) and (2.5) because the terms in $\partial^2 u^*/\partial \xi^2$ and $\partial^2 v^*/\partial \xi^2$ have entered on the right-hand sides, but on substitution of the form of solution corresponding to (2.7), $u^* = \xi u_0$, etc., we recover (2.3a)–(2.6a).

The term $\Psi_\eta u$ was also neglected by Kestin & Wood (1970), but they did not estimate the range of x for which this approximation is valid [$x = O(R^{-\frac{1}{2}})$].

The centrifugal terms are obviously important when $x = O(1)$ and to calculate their effect it is first necessary to study the region in which all those terms on the left of (2.11) are equal in importance, which turns out to be $x = O(R^{-\frac{1}{4}})$. This has not been pursued here as there is no reason to suppose that it will affect the question of stability; it will merely affect the downstream growth of the disturbance.

3. Numerical solution

Here we outline the numerical solution of the eigenvalue problem defined by (2.3a)–(2.6a) and (2.8). After extracting the real part and eliminating p_0 this becomes

$$\left.\begin{aligned} u_0+v_0'+\alpha w_0 &= 0,\\ u_0''+Fu_0'-(\alpha^2+\beta+2F')\,u_0 &= F''v_0,\\ \Omega_0''+F\Omega_0'-(\alpha^2+\beta-F')\,\Omega_0 &= 0,\\ \Omega_0 &= w_0'+\alpha v_0. \end{aligned}\right\} \tag{3.1}$$

We have the boundary conditions at $\eta = 0$ given by (2.9) and by considering the behaviour of (3.1) as $\eta \to \infty$ the boundary conditions there can be written as

$$\left.\begin{aligned} \Omega_0 &\sim A\zeta^{-\alpha^2-\beta}\exp(-\tfrac{1}{2}\zeta^2),\\ u_0 &\sim B\zeta^{-\alpha^2-\beta-3}\exp(-\tfrac{1}{2}\zeta^2),\\ v_0 &\sim C\exp(-\alpha\zeta)-\alpha A\zeta^{-\alpha^2-\beta-2}\exp(-\tfrac{1}{2}\zeta^2),\\ w_0 &\sim C\exp(-\alpha\zeta)-A\zeta^{-\alpha^2-\beta-1}\exp(-\tfrac{1}{2}\zeta^2), \end{aligned}\right\} \tag{3.2}$$

where $\zeta = \eta - 0{\cdot}64790$, this number being the result of the numerical integration of the equation for F, equation (2.2).

Prolonged attempts were made to solve this system by shooting methods and by finite differences, and were eventually abandoned. Although some progress had been made the computations were proving far too expensive. The problem was eventually solved using the ideas of invariant imbedding and Riccati transformations (see for example Scott 1973; Aziz 1973).

Here we introduce vectors $\mathbf{y}_1$ and $\mathbf{y}_2$ given by

$$\mathbf{y}_1^{\mathrm{T}} = (u_0, v_0, w_0), \quad \mathbf{y}_2^{\mathrm{T}} = (u_0', \Omega_0, \Omega_0'). \tag{3.3}$$

The system (3.1) can be written as a first-order system in these variables, taking the form

$$\mathbf{y}_1' = \mathbf{A}_1\mathbf{y}_1+\mathbf{A}_2\mathbf{y}_2, \quad \mathbf{y}_2' = \mathbf{A}_3\mathbf{y}_1+\mathbf{A}_4\mathbf{y}_2, \tag{3.4}$$

where $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$ and $\mathbf{A}_4$ are 3×3 matrices. Now let $\mathbf{y}_1 = \mathbf{R}\mathbf{y}_2$; substituting into (3.4) we find that $\mathbf{R}$ satisfies the equation

$$\mathbf{R}' = \mathbf{A}_1\mathbf{R}-\mathbf{R}\mathbf{A}_4+\mathbf{A}_2-\mathbf{R}\mathbf{A}_3\mathbf{R} \tag{3.5}$$

and it may be shown that the initial condition on $\mathbf{R}$ is

$$\mathbf{R}(0) = 0. \tag{3.6}$$

(The partitioning of the system into $\mathbf{y}_1$ and $\mathbf{y}_2$ was designed to produce this convenient starting value.)

It is now straightforward in principle to integrate (3.5) out to some suitably large value of η, and arrange a match with the conditions (3.2). However it is easy to show that $\mathbf{R}$ will contain exponentially large elements (which were observed numerically) and it is better to switch to the matrix $\mathbf{S} = \mathbf{R}^{-1}$, which contains no exponentially large elements and satisfies a similar equation to (3.5):

$$\mathbf{S}' = \mathbf{A}_4\mathbf{S}-\mathbf{S}\mathbf{A}_1+\mathbf{A}_2-\mathbf{S}\mathbf{A}_3\mathbf{S}. \tag{3.7}$$

α	β	α	β
0·005	−1·513	0·5	−1·433
0·1	−1·436	0·6	−1·501
0·2	−1·390	0·7	−1·595
0·298	−1·3754	0·8	−1·712
0·3	−1·376	0·9	−1·853
0·4	−1·390		

TABLE 1. Wavenumber α and corresponding growth rate β.

The matrix $\mathbf{S}$ is much cheaper to calculate than $\mathbf{R}$. The switch from $\mathbf{R}$ to $\mathbf{S}$ was made at $\eta = 1$.

In order to incorporate the boundary conditions at infinity into the scheme we write

$$\mathbf{y}_1 = \mathbf{M}_1\mathbf{a}, \quad \mathbf{y}_2 = \mathbf{M}_2\mathbf{a}, \tag{3.8}$$

where $\mathbf{a}^{\mathrm{T}} = (A, B, C)$ and $\mathbf{M}_1$ and $\mathbf{M}_2$ are 3×3 matrices whose elements are the exponentially small factors contained in (3.2). Then at infinity we have

$$\mathbf{M}_1\mathbf{a} = \mathbf{R}\mathbf{M}_2\mathbf{a}, \tag{3.9}$$

or equivalently,

$$\mathbf{S}\mathbf{M}_1\mathbf{a} = \mathbf{M}_2\mathbf{a}. \tag{3.10}$$

It is necessary to have non-trivial solutions for $\mathbf{a}$ and so the condition at infinity is

$$\det(\mathbf{S}\mathbf{M}_1 - \mathbf{M}_2) = 0. \tag{3.11}$$

The results are given in table 1 and it is believed that they are accurate to all the figures given. They were not particularly sensitive to the value chosen for infinity, usually $\eta = 6$, except for small values of α, when it was necessary to integrate out to $\eta = 10$ to obtain the required accuracy. The range $1 \leqslant \alpha \leqslant 10$ was investigated and the values of β obtained were always less than the value at $\alpha = 1$.

The possibility that there is another set of eigenvalues with positive values of β was checked by investigating the region $0 < \beta < 10$ for selected values of α, but none were found. There was some evidence of another set of values of β below those given here but this was not pursued in detail.

All calculations were performed on the University of Manchester Regional Computer Centre CDC 7600 computer. Details of the integration routines, etc. are available from the authors, and these and descriptions of the shooting methods and finite-difference methods are available as a University of Manchester Numerical Analysis Report, no. 18.

4. Conclusions

It has been shown that viscous stagnation-point flow is stable to infinitesimal disturbances, periodic in the direction normal to the plane of the flow, in the limit of infinite Reynolds number. The experimental evidence for instability therefore requires some discussion. There are three limitations on the theory which may account for the disagreement.

(*a*) Infinite Reynolds number. Since experiments are of course carried out at finite R (and in many cases not particularly large R), a correction might be thought necessary.

It is known that a parallel shear flow can be unstable (to two-dimensional disturbances) at finite R but stable at infinite R. However it is unlikely that this will be the case here; one may make a comparison with cylindrical Couette flow, which is unstable at all values of the Taylor number above the critical value.

(*b*) The theory is limited to the immediate neighbourhood of the stagnation line, in fact up to $x = O(R^{-\frac{1}{2}})$, which is about $\frac{1}{2}°$ for a circular cylinder in most cases. For practical reasons most measurements were taken a few degrees round the cylinder from the stagnation line, where the vortices had grown to sufficient strength. We may suppose that the wavelength (as distinct from the amplitude) is not greatly affected by this for the following reason. It is not difficult to see how the disturbance solution can be extended round the cylinder as a power series in x, using the known solution for the basic flow (which is also a power series in x). This process will generate systems of equations similar to (2.3*a*)–(2.6*a*) with non-zero right-hand sides; the left-hand sides will not, however, be identical because of the higher powers of x involved. Therefore the value of β already obtained is *not* an eigenvalue of the new homogeneous system and the existence of a solution for the higher terms is assured.

(*c*) Nonlinear effects. The experiments show a feature not evident in the theory, which is that the secondary vortices grow in amplitude away from the stagnation line, and presumably at some stage nonlinear effects will become important. It is possible that the least-damped waves, for which $\alpha = 0{\cdot}298$, would be unstable to finite disturbances, and the nonlinear effects would presumably also have some effect on the wavelength. If any such nonlinear theory were to yield a unique most-unstable wavenumber α, the theory would predict that the experimental points should lie on a straight line through the origin, having slope $\frac{1}{10}\pi\alpha^{-1}$. For the least-damped waves in the present theory this number is about 1·05, so that all observed waves have shorter wavelengths. The observations do in fact lie fairly close to a line having slope about 0·5, corresponding to $\alpha \simeq 0{\cdot}6$.

It seems more likely that the growth of the vortices away from the stagnation line is due to vortex stretching by the main flow as suggested by Kestin & Wood, but it is still necessary to explain how the secondary vortices originate near the stagnation line. It is unlikely that the secondary vortices are produced simply as a result of the amplification, by the stretching mechanism, of small vortices already present in the oncoming stream, and there are two reasons for this. First, in the experiments of Hassler and Colak-Antic the solid body was towed through a water tank in which the water was at rest, so that the 'free-stream' vorticity would have been exceedingly small. Second, the experimental results, summarized in figure 1, suggest strongly that the spacing of the secondary vortices is determined by the boundary-layer thickness; this in turn suggests that they are due to some definite boundary-layer instability.

An ambiguous feature of the experiments is the role of free-stream turbulence. The experiments of Kestin & Wood (1970) show that increasing the intensity of the turbulence reduces the wavelength, but Brun *et al.* (1966) found the reverse. The pair of points in the diagram taken from their results is not an isolated result; experiments were carried out with yawed cylinders and only the results for zero yaw angle are given here, but the effect of turbulence was the same at all angles.

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